

FAMILIES OF DIRAC OPERATORS AND QUANTUM AFFINE GROUPS

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Abstract Twisted K-theory classes over compact Lie groups can be realized as families of Fredholm operators using the representation theory of loop groups. In this talk I want to show how to deform the Fredholm family, in the sense of quantum groups. The family of Dirac type operators is parametrized by vectors in the adjoint module for a quantum affine algebra and transform covariantly under a (central extension of) the algebra.

1. INTRODUCTION

Let X be a topological space, $Fred_*$ the space of self-adjoint Fredholm operators in a complex Hilbert space H with both positive and negative essential spectrum. This is a universal classifying space for K^1 . Actually, one can take as the **definition**:

$$K^1(X) = \{\text{homotopy classes of maps } f : X \rightarrow Fred_*\}$$

In the even case

$$K^0(X) = \{\text{homotopy classes of maps } f : X \rightarrow Fred\}$$

where $Fred$ is the space of all Fredholm operators in H .

The Chern character

$$ch : K^1(X) \rightarrow H^{odd}(X, \mathbb{Z})$$

is an additive map to odd cohomology classes. In particular, the degree 3 component $DD(f) = ch_3(f)$ of $[f] \in K^1(X)$ is called the **Dixmier-Douady class** of the gerbe defined by the family $f(x)$ of Fredholm operators. In the de Rham cohomology an equivalent construction of $DD(f)$ comes from the family $L_{\lambda\lambda'}$ of complex line bundles. One can choose the curvature forms $\omega_{\lambda\lambda'}$ such that

$$\omega_{\lambda\lambda'} + \omega_{\lambda'\lambda''} = \omega_{\lambda\lambda''}$$

and with a partition of unity $\sum \rho_\lambda = 1$ subordinate to the cover by the open sets U_λ one has

$$DD(f) = \sum_{\lambda} d\rho_\lambda \wedge \omega_{\lambda\lambda'}$$

and this does not depend on the choice of λ' . The 3-cohomology class is related to quantum field theory anomalies and it transgresses in the Hamiltonian formulation of gauge theory to central, or more generally, abelian extension of the group of gauge transformations, [CMM].

A 3-cohomology class $[H]$ can be used as twisting in K-theory. Instead of a globally defined family of Fredholm operators over X we have local families $f_\alpha : U_\alpha \rightarrow \text{Fred}(H)$, where $\{U_\alpha\}$ is an open cover of X , such that on the overlaps $U_\alpha \cap U_\beta$ one has

$$f_\beta(x) = h_{\beta\alpha}(x) f_\alpha(x) h_{\beta\alpha}(x)^{-1}$$

where the $h_{\alpha\beta}$'s are the transition functions for a principal $PU(H)$ bundle P over X . The equivalence class of the bundle P is given by $[H]$. The local 2-forms $\omega_{\alpha\beta}$ are given as the pull-backs $h_{\alpha\beta}^* \omega$ where ω is the Chern class of the canonical complex line bundle over $PU(H)$.

Quantum field theory provides a method for constructing twisted families of Fredholm operators. In particular, one can use the supersymmetric Wess-Zumino-Witten model in the case when X is a compact Lie group. This construction actually provides an operator theoretic realization for the relation between twisted K-theory and the Verlinde algebra in conformal field theory, proven in [FHT]. In the next section I will briefly recall the construction of the undeformed family of Dirac operators, [M].

2. THE SUPERSYMMETRIC WZW MODEL

Families of Dirac operators D_A on the unit circle S^1 coupled to smooth vector potentials A transform covariantly under the gauge group action, defining an element in $K^*(\mathcal{A}/\mathcal{G}) = K^*(G)$. Here $\mathcal{G} = \Omega G$, the based smooth loop group of a compact Lie group G . The quantized Dirac operators \hat{D}_A acting in a fermionic Fock space transform covariantly under a central extension \widehat{LG} of the full loop group LG , thus the family \hat{D}_A is a candidate for an element in twisted K-theory on the moduli space of gauge connections. However, there is a catch: These operators are essentially positive, we need operators with both positive and negative essential spectrum.

To find genuine nontrivial twisted K-theory classes we have to go to the supersymmetric WZW model. Morally, the family of Fredholm operators is now a family of Dirac operators on the loop group LG . We cannot make sense of the Dirac operators on the

infinite-dimensional manifold LG by the standard analytic methods (we would need a Haar measure on the group LG) but there is a purely algebraic construction using the highest weight representation theory of \widehat{LG} .

Concretely, the operators are acting in a tensor product $H = H_f \otimes H_b$, where H_b carries an irreducible highest weight representation of \widehat{LG} of level $k = 0, 1, 2, \dots$ and H_f carries an irreducible representation of the Clifford algebra of the vector space $L\mathfrak{g}$. The Clifford algebra is then used to construct a (reducible) representation of the central extension $\hat{\mathfrak{g}}$ of $L\mathfrak{g}$, of level κ , the dual Coxeter number of \mathfrak{g} . A basis for the Clifford algebra consists of elements ψ_a^n with $n \in \mathbb{Z}$ and $a = 1, 2, \dots, \dim \mathfrak{g}$ with anticommutation relations

$$\psi_a^n \psi_b^m + \psi_b^m \psi_a^n = 2\delta_{ab}\delta_{n,-m}.$$

The commutation relations of the basis vectors T_a^n of $L\mathfrak{g}$ in the representation space H_b are

$$[T_a^n, T_b^m] = \lambda_{abc} T_c^{n+m} + \frac{k}{4} \delta_{ab} n \delta_{n,-m}$$

where the λ_{abc} 's are the structure constants of \mathfrak{g} in a suitably normalized basis and the Dirac operator Q_A coupled to a vector potential on S^1 is

$$\begin{aligned} (1) \quad Q_A &= i\psi_a^n \otimes T_a^{-n} - \frac{i}{12} \lambda_{abc} \psi_a^n \psi_b^m \psi_c^{-n-m} \otimes 1 + i \frac{k+\kappa}{4} \psi_a^n A_a^{-n} \otimes 1 \\ (2) \quad &= i\psi_a^n \otimes T_a^{-n} + \frac{i}{3} \psi_a^n K_a^{-n} \otimes 1 + i \frac{k+\kappa}{4} \psi_a^n A_a^{-n} \otimes 1, \end{aligned}$$

where K_a^n 's satisfy the same commutation relations, as operators in H_f , as the operators T_a^n except that the level is κ instead of k . Here A_a^n 's are the Fourier components of a vector potential on the circle.

The family Q_A transforms covariantly under the projective representation of level $k + \kappa$ the loop group LG defining an element in $K(G, k + \kappa)$ corresponding to the D-D class $[H]$ in $H^3(G, \mathbb{Z})$ equal to $k + \kappa$ times the basic class in $H^3(G) = \mathbb{Z}$ when G is a simple simply connected compact Lie group.

Actually, since $\mathcal{A}/\Omega G = G$ and $G \subset LG$, we have an G equivariant class, element of $K_G^*(G, H)$. In the q-deformed case the gauge symmetry is not a group but a Hopf algebra, and the moduli space of gauge connections is not defined. Instead, the role of the Dixmier-Douady class $[H]$ is completely taken over by the (level of) the central extension of the quantum affine algebra.

Our construction can be viewed as a generalization of [BK], from the quantum $SU(2)$ algebra to quantum affine algebras. A more detailed article is in preparation, [HM]. I want to stress that we are not trying to construct a spectral triple, in the sense of noncommutative geometry, using the generalized Dirac operator. Such constructions

for compact quantum groups have been discussed in several papers during the last ten years. I just refer to one recent paper [NT], see for references to earlier literature therein.

3. QUANTUM AFFINE ALGEBRA

Let \mathfrak{g} be a simple finite-dimensional Lie algebra and $\hat{\mathfrak{g}}$ the associated affine Lie algebra. The quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is generated by vectors $e_0, e_1, \dots, e_\ell, f_0, f_1, \dots, f_\ell, K_0, K_1, \dots, K_\ell, K_0^{-1}, \dots, K_\ell^{-1}$ with the relations

$$\begin{aligned} [e_i, f_i] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, K_i K_j = K_j K_i \\ K_i e_j K_i^{-1} &= q^{\alpha_{ij}} e_j, K_i f_j K_i^{-1} = q^{-\alpha_{ij}} f_j \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q e_i^{1-a_{ij}-k} e_j e_i^k &= 0 \quad (i \neq j) \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_q f_i^{1-a_{ij}-k} f_j f_i^k &= 0 \quad (i \neq j) \end{aligned}$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{m_q(m-1)_q \dots (m-k+1)_q}{k_q(k-1)_q \dots 1_q}$$

$$k_q = 1 + q + \dots q^{k-1}.$$

Here the parameter q is a positive real number and the integers a_{ij} are the matrix elements of the Cartan matrix of $\hat{\mathfrak{g}}$.

Let A_i^n with $n \in \mathbb{Z}$ and i is a label for the basis in the adjoint module of \mathfrak{g} , be a basis for the q -affine adjoint module. Under \mathfrak{g} each 'Fourier mode' A^n transforms according to the adjoint representation of $U_q(\mathfrak{g})$, which is a q -deformation of the adjoint representation of \mathfrak{g} . The generator e_0 increases the index n by one unit, f_0 decreases it by one unit. For example, for $\mathfrak{g} = \mathfrak{sl}(2)$ one has the explicit formulas

$$\begin{aligned}
e_1 A_1^n &= f_0 A_1^n = 0, & f_1 A_1^n &= A_0^n, & e_0 A_1^n &= A_0^{n+1} \\
e_1 A_0^n &= (q + q^{-1}) A_1^n, & f_0 A_0^n &= (q + q^{-1}) A_1^{n-1} \\
f_1 A_0^n &= A_{-1}^n, & e_0 A_0^n &= A_{-1}^{n+1} \\
e_1 A_{-1}^n &= (q + q^{-1}) A_0^n, & f_0 A_{-1}^n &= (q + q^{-1}) A_0^{n-1} \\
f_1 A_{-1}^n &= 0 = e_0 A_{-1}^n \\
K_1 A_i^n &= q^{2i} A_i^n = K_0^{-1} A_i^n.
\end{aligned}$$

The vectors A_i^n will be constructed as operators acting in a Fock space carrying a representation of $U_q(\hat{\mathfrak{g}})$ such that the adjoint action is given by

$$x.A_i^n = \sum_{(x)} x' A_i^n S(x'') \text{ for } x \in U_q(\hat{\mathfrak{g}}),$$

where $S : U_q(\hat{\mathfrak{g}}) \rightarrow U_q(\hat{\mathfrak{g}})$ is the antipode and $\Delta(x) = \sum_{(x)} x' \otimes x''$ is the coproduct $\Delta : U_q \rightarrow U_q \otimes U_q$. We also need the Clifford algebra generated by elements ψ_i^n acting in the Fock space and transforming under $U_q(\hat{\mathfrak{g}})$ according to the dual adjoint representation (which in fact is equivalent to the adjoint representation).

4. THE DIRAC OPERATOR

The Dirac operator Q is acting in $H_f \otimes H_b$ where H_f is the q-fermionic Fock space and H_b carries another highest weight representation of $U_q(\hat{\mathfrak{g}})$. The action of the nontrivial central extension is seen in the action of the element $K_0 K_1 \dots K_\ell$, which is no more equal to the unit operator but a power of q where the exponent depends on the level of the representation.

$$Q = i \sum \psi_a^n \otimes T_a^{-n} + i \frac{1}{3} \sum \psi_a^n K_a^{-n} \otimes 1$$

where T_a^n are basis vectors of the adjoint module, acting as linear operators in the space H_b . We need also another copy of the adjoint module, acting in the space H_f . The components are denoted by K_a^n . The vectors ψ_a^n are elements in a quantum Clifford algebra acting as operators in a q-Fock space H_f , to be defined below in Subsection 4.1.

In contrast to the undeformed case, the operators K_a^n, T_a^n do not satisfy the defining relations of the algebra $U_q(\hat{\mathfrak{g}})$.

Let R be the universal R-matrix for the algebra U_q . An explicit construction is given in [KT]. Following [DG], we can then define a basis for vectors in a submodule $A \subset U_q$

transforming according to an adjoint representation

$$ad_q(x)v = \sum_{(x)} x'vS(x'')$$

of U_q on itself. A basis is defined as

$$A_i^n = \sum K_{n,i}^{m,\alpha;p,\beta} (\pi_{m,\alpha;p,\beta} \otimes id) A,$$

where $A = (R^T R - 1)/h$, with $e^h = q$ and $R^T = \sigma R \sigma$, where σ permutes the factors in the tensor product $U_q \otimes U_q$. Here $\pi_{m,\alpha;p,\beta}$ are the matrix elements in the defining representation V of U_q .

For example, for $\hat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(2)$ the basis in the defining representation is v_i^n with $n \in \mathbb{Z}$ and $i = -1, 0, 1$ and $\alpha, \beta = \pm$. The numerical coefficients K come from the identification of the basis of the adjoint representation as linear combinations of the basis vectors in $V \otimes V$.

The action of the Serre generators in the defining representation is

$$\begin{aligned} e_1 v_+^n &= f_0 v_+^n = 0, f_1 v_+^n = v_-^n, e_0 v_+^n = v_-^{n+1}, e_0 v_+^n = v_-^{n-1} \\ e_1 v_-^n &= v_+^n, f_0 v_-^n = v_+^{n-1}, e_0 v_-^n = 0 = f_1 v_-^n \\ K_1 v_\pm^n &= q^{\pm 1} v_\pm^n = K_0^{-1} v_\pm^n. \end{aligned}$$

4.1. Generalized affine Hecke algebra, the case of $U_q(\widehat{\mathfrak{sl}}(2))$. The affine Hecke algebra for $\hat{\mathfrak{g}}$ is defined through the relations coming from the R-matrix $\check{R} = \sigma R$ in the tensor product $V^0 \otimes V^0$. See [L] for the use of the Hecke algebra for constructing Fock space representations of $U_q(\widehat{\mathfrak{sl}}(n))$. The matrix satisfies

$$(\check{R} - q^{-1})(\check{R} + q) = 0,$$

since $-q$ and q^{-1} are the only eigenvalues of the invertible matrix \check{R} . Denote by Y_1 the shift operator which sends $v_i^n \otimes v_j^m$ to $v_i^{n+1} \otimes v_j^m$ and by Y_2 the corresponding shift operator acting on the second tensor factor. The matrix \check{R} acting on V is then defined using the relations

$$\check{R}Y_1 = Y_2\check{R}^{-1}, \quad \check{R}Y_2 = Y_1\check{R} + (q - q^{-1})Y_2.$$

Actually, the second relation follows from the first and the minimal polynomial relation.

Now the braiding relations are given by setting the ideal in the tensor algebra of V generated by the elements

$$(q^{-1} + \check{R})(V \otimes V)$$

equal to zero. These have in particular the consequence that any $v_i^n v_j^m$ with $n > m$ can be written as a linear combination of vectors $v_k^p v_l^q$ with $p + q = n + m$ and $p \leq q$. In the

zero mode space V^0 the meaning of the braiding relations is that they project out the 'symmetric' part of the tensor product $V^0 \otimes V^0$. The 3-dimensional representation is the eigenspace of \check{R} with eigenvalue q^{-1} and the 1-dimensional component corresponds to the eigenvalue $-q$.

To complete the construction of the Dirac operator we need also the generalized Clifford algebra in the coadjoint representation. The algebra is generated by vectors ψ_i^n with $n \in \mathbb{Z}$ and $i = 1, 0, -1$. The defining relations are given by braiding relations and an invariant (nonsymmetric) bilinear form. The braiding relations are defined recursively like in the case of V (or V^*) with the difference that since the R -matrix \check{R} in the adjoint representation has 3 instead of 2 different eigenvalues, which are now $-q^{-2}, q^2, q^{-4}$, with multiplicities 3, 5, 1 respectively. The Fock space representation of the Clifford algebra is defined similarly to the undeformed case: It is generated by a finite-dimensional irreducible spin representation of the zero Fourier mode algebra, with the additional relation that this subspace is annihilated by the elements ψ_i^n for $n < 0$.

Finally, the action of $U_q(\widehat{\mathfrak{sl}}(n))$ is defined in the tensor algebra modulo the ideal generated by the elements (3) below, over the coadjoint module, using the opposite coproduct $\Delta^{op}(x) = \sum_{(x)} x'' \otimes x'$.

The negative eigenvalue corresponds again to a 3-dimensional 'antisymmetric' representation and the positive eigenvalues to a 6-dimensional 'symmetric' representation; the latter contains the 1-dimensional trivial representation.

The Hecke algebra is replaced by a generalized Hecke algebra,

$$\begin{aligned} Y_1 Y_2 &= Y_2 Y_1 \\ (\check{R} - q^2)(\check{R} - q^{-4})(\check{R} + q^{-2}) &= 0 \\ \check{R} Y_1 &= Y_2 \check{R}^{-1}, \quad \check{R} Y_2 = Y_1 \check{R} + (q^2 - q^{-2}) Y_2 \end{aligned}$$

where the middle relation is the minimal polynomial of the diagonalizable matrix \check{R} .

The generalized symmetric tensors correspond to positive eigenvalues of \check{R} . In the Clifford algebra symmetrized products are identified as scalars times the unit. That is, we fix a $U_q(\widehat{\mathfrak{sl}}(2))$ invariant bilinear form B and the Clifford algebra is defined as the tensor algebra over V modulo the ideal generated by

$$(3) \quad P(u \otimes v) - B(u, v) \cdot 1$$

where P is the projection on positive spectral subspace of \check{R} . In the case when V is the adjoint module for $U_q(\widehat{\mathfrak{sl}}(2))$ one can fix B by identifying the first factor V as the dual

V^* and using the natural pairing $V^* \otimes V \rightarrow \mathbb{C}$. Alternatively, one can view B as the projection onto the 1-dimensional trivial submodule inside of the 'symmetric module'.

4.2. The family of Dirac operators. In the nondeformed case one has for an infinitesimal gauge transformation $X \in L\mathfrak{g}$

$$[X, Q] = (k + \kappa) \sum (-n) \psi_i^n X_i^{-n} = \frac{k + \kappa}{4} \langle \psi, dX \rangle$$

and for a family of operators $Q_A = Q + \frac{k+\kappa}{4} \psi_i^n A_i^{-n}$

$$[X, Q_A] = \frac{k + \kappa}{4} \langle \psi, [A, X] + dX \rangle.$$

In q-deformed case A is to be understood as a vector in the adjoint module extended by $\mathbb{C}c$. The above equation is replaced by

$$\sum_{(x)} x' Q_A S(x'') = Q_{x \cdot c A}$$

where x_c denotes the adjoint action in the centrally extended module,

$$x \cdot c A = x \cdot A + \lambda_x(A) c$$

with λ_x a linear form on the adjoint module, linear in the argument $x \in U_q(\widehat{\mathfrak{sl}}(2))$. It satisfies the cocycle relation

$$\lambda_{xy}(A) = \lambda_x(y \cdot A).$$

Here c is an element of the extended module such that $x \cdot c = 0$ for all $x \in U_q(\widehat{\mathfrak{sl}}(2))$. It is enough to give value of λ_x when x is a Serre generator. The only nonzero forms are λ_{e_0} and λ_{f_0} . The former is nonzero only for the component $A = A_1^{-1}$ and the latter for $A = A_{-1}^1$. In the particular case when the adjoint module is acting as operators in a level = 1 representation of $U_q(\widehat{\mathfrak{sl}}(2))$ we have $\lambda_{e_0}(A_1^{-1}) = -q^{-1}$ and $\lambda_{f_0}(A_{-1}^1) = q^{-1}$.

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